Fundamentals of Stochastic Processes 048868

Home Assignment 3, 4 May 2009

Martingales

Submit by 18 May

1. Let $\xi_1, \xi_2, \ldots$ be independent with $E\xi_i = 0$ and $E\xi_i^2 = \sigma_i^2$. Let $S_n = \sum_1^n \xi_i$, $s_n^2 = \sum_1^n \sigma_i^2$. Show that $S_n^2 - s_n^2$ is a martingale.

2. Let X_n be a submartingale. Show that it is a martingale if and only if $EX_n = EX_0$ for all n.

3. Let $Y_1, Y_2, \ldots,$ be nonnegative i.i.d. random variables with $EY_1 = 1$. Let $X_m = \prod_{n \le m} Y_n$.

   (a) Show that X_m is a martingale.

   (b) Assume that $P(Y_1 = 1) < 1$. Use the positive supermartingale convergence theorem, and an argument by contradiction to show that $X_m \to 0$ a.s.

   (c) Use the strong law of large numbers on $\log Y_n$ to arrive at the same conclusion.

4. Give an example of a martingale X_n with $X_n \to -\infty$ a.s. Hint: Let $X_n = \xi_1 + \ldots + \xi_n$, where the ξ_i are independent (but not identically distributed) with $E\xi_i = 0$.

5. Let X_n be a martingale with $X_0 = 0$ and $E(X_n)^2 < \infty$. Using the fact that $(X_n + c)^2$ is a submartingale and optimizing over c, show that

$$P\{\max_{1 \le m \le n} X_m \ge \lambda\} \le EX_n^2/(EX_n^2 + \lambda^2).$$