Hash Tables With Finite Buckets Are Less Resistant to Deletions

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Abstract

We show that when memory is bounded, i.e. memory buckets are finite, dynamic hash tables that allow insertions and deletions behave significantly worse than their static counterparts that only allow insertions. This behavior differs from previous results in which, when memory is unbounded, the two models behave similarly.

We show the decrease in performance in dynamic hash tables using several hash-table schemes. We also provide tight upper and lower bounds on the achievable overflow fractions in these schemes. Finally, we propose an architecture with content-addressable memory (CAM), which mitigates this decrease in performance.

Keywords: High-Speed Networks, Queuing Theory and Analysis, Dynamic Hash Tables

1. Introduction

1.1. Background

Networking devices often use *dynamic* hash tables, in which elements keep being both inserted and deleted as packets arrive and depart. Therefore, they differ from *static* ones, which are built only once. However, for simplicity, device designers typically model the performance of the dynamic hash tables

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using models of the static hash tables. This paper shows that these static models can lead to a significant *under-estimation of the drop rate* in the dynamic case.

This underestimation of the drop rate can potentially affect the performance of networking devices. Hash tables form the core building block of many networking device operations, such as flow counter management, flow state keeping, elephant traps, virus signature scanning, and IP address lookup algorithms [1]. If memory is allocated to the dynamic hash tables according to the static model, more elements might need to be dropped from the hash tables than initially estimated.

Using the static model seems natural. Dynamic hash tables are known for being *typically harder to model* than static ones, sometimes even lacking any mathematical analysis [1]. Therefore, the static model appears to be a simpler and more accessible option to the network designer.

More significantly, former studies have also found the *same asymptotic behavior* in dynamic and in static hash tables, in at least three cases. These studies considered both the static case in which n elements are only inserted into n buckets of infinite size, and a specific dynamic model, in which a fixed load of n elements is kept by alternating between a randomly chosen element deletion and a new element insertion:

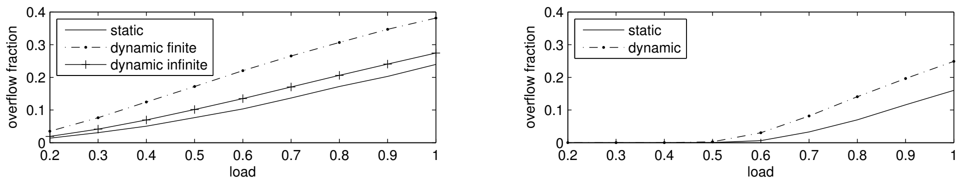
(a) In case the elements are *uniformly* hashed into the buckets, the maximum bucket size is known to be approximately $\log n/\log\log n$ with high probability. The dynamic model yields the same result [2, 3].

(b) Likewise, when inserting each element in the least-loaded of two random buckets (d-random algorithm with $d = 2$), the maximum bucket size is $\log\log n/\log 2 + O(1)$ in the static case; and again, the dynamic case yields the same result [3, 4].

(c) Similarly, using the asymmetric d-left algorithm [5], the static case and the dynamic case yield again the same bound on the maximum bucket size [6].

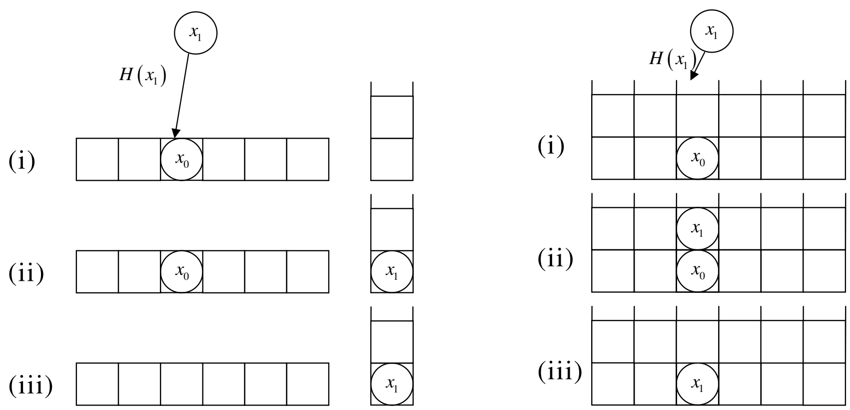
Therefore, as illustrated in these three cases, given a large number of elements, it appears that the network designer could use the simpler static model for the dynamic case.

In this paper, we focus on the realistic scenario in which buckets are finite, as used in networking devices, contrarily to the infinite-bucket case assumed above. We show that the dynamic hash table can exhibit a *significantly worse* drop rate than its static counterpart. That is, the rate of elements that cannot be inserted in the hash table is significantly higher.



(a) d-random with a stash (b) Cuckoo hashing with a stash

Figure 1: Average overflow fraction with 2 hash functions and bucket size 1, using both the static and the dynamic model.



(a) Finite buckets (b) Infinite buckets

Figure 2: An example demonstrating the degradation of performance in dynamic hash tables.

1.2. Performance Degradation

Fig. 1 plots the system average overflow fraction as a function of the load, i.e. the fraction of elements not placed in the buckets as a function of the average number of elements (either dropped or not dropped) per bucket [1]. Specifically, it shows the average overflow fraction for both a static system, where there are only insertions, and a dynamic system, where we alternate between deletions and insertions, while a fixed load is maintained [4, 7]. To measure the overflow fraction, it relies on an overflow list, called *stash*, to which new elements are moved when they cannot be inserted in the hash ta-

[1]Since in practice, sizing the buckets is according to the width of a single SRAM or DRAM memory word [1], we focus in this paper on buckets of small sizes (e.g. 1, 2 or 4). However, the results in this paper are general and consider any arbitrary bucket size.

ble. Fig. 1(a) and 1(b) show the overflow fraction of the d-random algorithm with a stash [4] (where $d = 2$), and the cuckoo hashing with a stash [8, 9]. The overflow fractions are obtained in simulations using 2048 buckets, 10^6 rounds with one random element deletion and one element insertion in each round (for the dynamic case), and a standard pseudorandom number generator to obtain hash values[2]. The overflow fraction in the static case is only measured at the end of all element insertions, while in the dynamic case it is measured after each cycle and then averaged over cycles.

Both figures clearly show a non-negligible degradation in the overflow fraction of the dynamic system. For instance, the cuckoo hashing scheme with load of 0.6 yields an overflow fraction of 0.62% and 3.02% in the static and dynamic models, respectively. Moreover, while for cuckoo hashing scheme with load of 0.5 the overflow fraction in the static model quickly goes to 0 [10], it does so more slowly in the dynamic case. For instance, for $m = 1024$ we get an overflow fraction in the static and dynamic models of 0.05% and 0.44%, where for $m = 16384$ we get 0.0012% and 0.0606%, respectively.

In the infinite-bucket case, no overflow list is needed. Thus, to compare the finite-bucket case with the case where buckets are infinite, we define the average overflow fraction for the infinite-bucket case as the probability that an element is not the first one in its bucket. Furthermore, since cuckoo hashing scheme is not defined for infinite buckets, we compare the overflow fraction only under the d-random scheme. Our simulations show a similar overflow fraction in the following three cases: static scheme with buckets of size one, static scheme with infinite buckets, and dynamic scheme with infinite buckets (the first two curves completely coincide in Fig. 1(a), so they appear in the legend as "static", while the last curve is slightly different due to the nature of the d-random scheme). On the other hand, there is a significant increase in the overflow fraction when the scheme is dynamic and buckets are finite. In the rest of the paper, we will evaluate this performance degradation and propose methods to mitigate this problem.

1.3. Intuition

The intuition behind this difference in behavior is that if the bucket size is bounded, once an element is placed in the overflow list it stays there regardless of whether the corresponding bucket become available later upon

[2]Simulations with ten times more buckets or rounds yielded nearly identical results.

deletion. Therefore, the order of the insertion and deletion operations directly affects the performance. This is typically not the case in the unbounded bucket case, and the difference can cause a drastic degradation in the scheme performance.

Fig. 2 illustrates this degradation in performance, using the same scenario both for the finite and the infinite bucket sizes. For the case of finite buckets, we assume bucket sizes of 1, an overflow list, and an insertion algorithm that uses only one hash function. We consider the following scenario: Let t be the time when a new element x_1 is hashed to a full bucket j that already stores element x_0 (step (i) in both Fig. 2(a) and Fig. 2(b)). If a finite bucket is used, then x_1 is moved to the overflow list (step (ii) in Fig. 2(a)), while in the infinite-bucket case, x_1 is simply stored in bucket j (step (ii) in Fig. 2(b)). Let $t' > t$ be the time when element x_0 is deleted. Assuming that element x_1 is not deleted before t', it stays in the overflow list in the finite-bucket case, while in the infinite-bucket case it is stored in bucket j (step (iii)).

Therefore, in the dynamic case with finite bucket sizes, element x_1 is in the overflow list, even though its corresponding bucket j is empty. This could never happen in the static case (elements are stored in the overflow list only after their corresponding buckets are full, and full buckets cannot become empty). It could also never happen in the dynamic case with infinite buckets (there is no overflow list).

1.4. Our Contributions

We start by considering a simplistic dynamic scheme with a single hash function. We model this hashing scheme analytically using two different models: a discrete-time Markov chain, and a fluid model with a continuous-time Markov process. We find that this simplistic dynamic scheme performs notably worse than its corresponding static scheme.

Then, we obtain a lower bound on the expected overflow fraction in the dynamic model of *any* hash-table scheme that uses uniform hash functions. We prove that when the *average* number of memory accesses per insertion a increases, the expected overflow fraction can decrease as slowly as $\Omega(1/a)$ (compared to $\Omega(e^{-a})$ in the static case [11]). This indicates that the poor performance of dynamic schemes is fundamental, and is hard to solve by simply using additional memory accesses (or hash functions).

Next, we introduce an online multiple-choice scheme (that is, a scheme that uses multiple hash functions). We demonstrate that this scheme reaches the lower bound and therefore is optimal up to a certain rate of memory access, which depends on the system parameters.

However, due to the slow decrease of the lower bound, optimality may be insufficient for certain applications. Therefore, we suggest changing the assumptions and moving back elements from the overflow list when a bucket becomes available upon deletion. We propose the M-B (Moving-Back) scheme that uses a CAM (Content-Addressable Memory) device that stores the elements along with their hash values. A parallel lookup operation is used once an element is deleted and its bucket becomes non-full. This operation, supported by the CAM, finds an element in the overflow list that can be moved back to the bucket. This scheme is shown to beat the initial lower bound without a CAM.

Finally, we evaluate all proposed schemes using simulations as well as experiments with real hash functions applied on real-life traces.

Paper Organization. We start with preliminary definitions in Section 2. Section 3 presents and analyzes the single-choice SINGLE scheme, while Sections 4 and 5 provide a lower bound on the expected overflow fraction. Then, in Section 6 we present and analyze the multiple-choice MULTIPLE scheme, and in Section 7 we present the CAM-based M-B scheme which, upon deletion, moves back elements from the overflow list. Finally, we evaluate all the analytical results in Section 8.

For the sake of readability, some proofs are presented in Appendix A. Appendix B provides additional explanations.

2. Problem Statement

2.1. Terminology and Notations

This paper considers *single- and multiple-choice hash schemes with a stash* [12, 11]. Such schemes consist of two data structures: *(i)* A *hash table* of total memory size $m \cdot h$, partitioned into m buckets of size h; *(ii)* An *overflow list*, usually stored in a CAM. Note that the overflow list can also be absent, in which case overflow elements are simply dropped.

Like traditional hash tables, the schemes should support three basic operations: element insertions, element deletions, and lookups. We call the (infinitely long) sequence of these operations the *input sequence* of the scheme. Consequently, we sometimes refer to an insertion operation as an arrival of an element, and to a deletion operation as a departure of an element. In this paper, we focus mostly on a specific input sequence, alternating between departures of a random element (picked uniformly at random) and arrivals of a new element [4, 7].

Multiple-choice hashing schemes employ up to d probability distributions over the set of buckets; these distributions are then used to generate a *hash-function set* $\mathcal{H} = \{H_1, \ldots, H_d\}$ of d independent hash functions. For each element x and each operation, the scheme can consider only the buckets $\{H_1(x), \ldots, H_d(x)\}$ (and the overflow list). In addition, we assume that the scheme must access a bucket to obtain any information on it.

In some cases we will make use of uniformly-distributed hash functions, as defined below.

Definition 1. *A* uniformly-distributed hash function *is a hash function that maps into each bucket with equal probability.*

In this paper, we focus on the average case behavior of the system, as formally defined below.

Definition 2. *The* expected overflow fraction *is the expected fraction of elements (over time) that are not placed in the buckets, that is, are either placed in the overflow list or simply dropped in case the overflow list is absent.*

Our goal is to minimize the *expected overflow fraction* of the scheme, subject to the (total and average) number of *memory accesses*. We count as one memory access reading and updating all the elements of a single bucket. This corresponds to the common practice of sizing the bucket size by the width of the memory word. We do not count accesses to the overflow list. We further assume that up to d buckets can be read in parallel before deciding which one to update, requiring a total of d memory accesses.

Formally, the hashing scheme and the optimization problem are captured by the following two definitions, where the load c is the ratio of the total number of elements n by the total memory size mh: $c = \frac{n}{mh}$.

Definition 3. *When the load is c and the bucket size is h, an $\langle a, d, c, h\rangle$* hashing scheme *is a scheme with an expected (respectively, maximum) number of memory accesses per element of at most a (respectively, d).*

Definition 4. *The* OPTIMAL DYNAMIC HASH TABLE PROBLEM *is to find an $\langle a, d, c, h\rangle$ hashing scheme that minimizes the expected overflow fraction γ as the number of elements n goes to infinity. Whenever defined, let γ_{OPT} denote this optimal expected limit overflow fraction.*

2.2. Input Models

Throughout the paper, we will use two different models for the arrivals and departures of elements: a *discrete finite model* with a finite number of elements; and a *fluid model* based on differential equations with an infinite number of elements. Our objective is to model a constant load, i.e. a constant number of elements in the system, so that departing elements are replaced by arriving elements.

In both of the models we start at time $t = 0$ with all the n elements placed in the overflow list. The description of the differences between the models is given below.

Discrete Finite Model — In the *discrete finite model*, we assume that time is divided into time-slots of unit duration. At the start of each time-slot $t > 0$, an element is chosen uniformly at random among all n elements in the system to depart. Next, at the end of time-slot t, a new element arrives and is inserted according to the hashing scheme into either a non-full bucket or the overflow list. Therefore, by the end of each time-slot t, there are always n elements in the system, either in the hash table or in the overflow list.

Fluid Model — The second model is the *fluid model*, which attempts to model the behavior of the continuous system as both the number of elements n and the number of buckets m go to infinity with a constant limit ratio $ch = \lim_{n\to\infty} \frac{n}{m}$. In the fluid model, we will often analyze the system using differential equations, and will be mainly interested in their fixed-point solutions.

In this model, each element stays in the system for an exponentially-distributed duration of average 1. Therefore, at each infinitesimal time-interval $[t, t + \delta t]$, the probability that a given element departs is $n \cdot \delta t + o(\delta t)$. As such, the departure rate from each bucket is proportional to the bucket size.

For each element departure, another element is automatically generated and inserted in the system. Thus, the average arrival rate per bucket is $\frac{n}{m}$, since the arrival rate is n and there are m buckets. Therefore, in the fluid model, we model a constant average arrival rate per bucket of $ch = \lim_{n\to\infty} \frac{n}{m}$.

Consider a finite number of buckets. When arriving elements use a uniformly-distributed hash function, they hash into each bucket at a rate equal to the average rate of $\frac{n}{m}$. However, since we consider in the fluid model an infinite number of buckets, the uniformly-distributed hash function translates to a *continuous* uniformly-distributed hash function. By extension, and for simplicity, we will define such a function as one that enables the same arrival rate of ch to all buckets.

Furthermore, we will define the average number of memory accesses per element a such that it is valid at any time t, thus we call it the *hashing rate*. We will also assume that the hash values are independent from the bucket occupancy, i.e. that the hashing rate of a is valid given any bucket size.

Model Alternatives — In general, to model system scaling, we would be interested in using the discrete finite model, and then in studying how its solution scales with n. However, given the complex interactions between the n elements, this model often prove intractable. Therefore, we will use the fluid model in these cases, and will most often *not be able* to prove convergence of the discrete finite model to the fluid model. Likewise, we will not always prove convergence of the differential equations to the fixed-point solutions. This is, of course, a limit of our analysis.

On the other hand, for the single-choice hashing scheme (Section 3), we provide a full analysis with both models, and prove that the limit of the discrete finite model behaves indeed like in the fluid model. In simulations, we will also show that the scaled systems converge fast to their fluid model. We refer to [13] for a more complete discussion of the sufficient conditions for the convergence to the fluid-limit fixed-point solution.

3. A Single-Choice Hashing Scheme

We start by analyzing a simplistic hashing scheme, which uses only a single uniformly-distributed hash function H to insert elements in the hash table. Each element x is stored in bucket $H(x)$, if it is not full, and in the overflow list otherwise. Since an element uses exactly one hash function, its average number of memory accesses per element is $a = 1$. Of course, this simplistic scheme would probably not be implemented in advanced networking devices. However, it provides a better intuition on the reasons behind the performance degradation in dynamic hash-table schemes.

Discrete Finite Model — We first develop an analytical model for the scheme within the discrete framework presented in Section 2. Let $p_k(t)$ denote the *expected* fraction of buckets that have k elements at the end of time-slot t, and $p(t) = (p_0(t), p_1(t), \ldots, p_h(t))$. Using the discrete finite model, we obtain the following result on the limits of $p(t)$ and of the expected overflow fraction. The full proof appears in Appendix A.1 and is based on a birth-death Markov chain that models the occupancy of an arbitrary bucket over time.

Theorem 1. *Let* $C = \sum_{\ell=0}^{h} \binom{n}{\ell} (\frac{1}{m-1})^{\ell}$. *In the discrete finite model, when* $t \to \infty$,
(i) $p(t)$ *converges to the Engset distribution* π^n *[14, 15]; that is, for all* $0 \leq k \leq h$, $p_k(t)$ *converges to* π_k^n, *where*

$$\pi_k^n = \frac{1}{C} \cdot \binom{n}{k} \cdot \left(\frac{1}{m-1}\right)^k. \tag{1}$$

(ii) *the expected overflow fraction converges to*

$$\frac{1}{C} \cdot \binom{n}{h} \cdot \left(\frac{1}{m-1}\right)^h \cdot \left(1 - \frac{h}{n}\right). \tag{2}$$

Equation (1) can be rewritten as a truncated binomial expression

$$\pi_k^n = \frac{\binom{n}{k} \left(\frac{1}{m}\right)^k \left(1 - \frac{1}{m}\right)^{n-k}}{\sum_{l=0}^{h} \binom{n}{l} \left(\frac{1}{m}\right)^l \left(1 - \frac{1}{m}\right)^{n-l}}. \tag{3}$$

Since the proof of Theorem 1 shows that the distribution of a bucket occupancy also follows π_k^n (as $p(t)$ does), the above expression hints at the following interesting equivalent system: the bucket occupancy is distributed as if the n elements were assigned uniformly at random among the m buckets, and then the buckets with more than h elements were completely cleared out and had *all* their elements put in the overflow list. This is in contrast with the static system in which only elements exceeding the bucket capacity of h are placed in the overflow list. Therefore, Equation (3) nicely illustrates the difference between the static and dynamic cases.

A detailed example of the behavior of the scheme in the dynamic and static setting appears in Appendix B.1, which shows a simplistic setting where, as the number of buckets increases (with fixed load), the dynamic case yields an expected overflow fraction of 50%, while the static case has an expected overflow fraction of only $e^{-1} \approx 36.79\%$.

Fluid Model — We now analyze the *infinite* system using a fluid model. In the fluid model, elements stay in the system for an exponentially-distributed duration of average 1, and therefore the departure rate from each bucket is proportional to the bucket size. In addition, when an element departs, a new element is inserted into the hash table (or in the overflow list if the corresponding bucket is full). As explained in Section 2, the arrival rate to each bucket is therefore $ch = \lim_{n\to\infty} \frac{n}{m}$.

The following theorem, which is based on the $M/M/h/h$ continuous-time Markov process [15], shows the performance of the scheme under the fluid model. (The full proof is in Appendix A.2).

Theorem 2. *In the fluid model,*
(i) $p(t)$ *converges to the stationary distribution* π^∞*, where*

$$\pi_k^\infty = \frac{(ch)^k}{k!} \Big/ \sum_{l=0}^{h} \frac{(ch)^l}{l!}, \quad k = 0, \ldots, h. \tag{4}$$

(ii) *the expected overflow fraction converges to* π_h^∞ *and follows the Erlang-B formula.*

We have seen that the discrete finite model with n elements yield a stationary distribution π^n, while the fluid model yields the distribution π^∞ (from the fixed-point equations). We will now show that as expected, when scaling n to infinity, π^n converges to π^∞, and so does the associated overflow fraction. (Proof in Appendix A.3.)

Corollary 3. *When* $n \to \infty$ *with* $\frac{n}{m} \to ch$*,*
(i) *the stationary distribution converges to the fixed-point distribution of the fluid model:* $\pi^n \to \pi^\infty$*; and*
(ii) *the expected overflow fraction of the discrete finite model converges to the overflow fraction of the fluid model.*

Finally, we generalize the scheme to deal with *probabilistic insertions*. Namely, there exists some $\alpha \in [0, 1]$ such that each arriving element is either hashed into a bucket as before with probability α, or placed directly in the overflow list with probability $1 - \alpha$, yielding an average number of memory accesses α (or equivalently, a total number of memory accesses $\alpha n \leq n$, less than the number of elements). Using the fluid model for simplicity, we obtain the following result. While this probabilistic scheme is probably not useful in practice (since the average memory access rate is seldom less than 1), we will later demonstrate that it is *optimal* under specific conditions.

Theorem 4. *In the fluid model, given the single-choice hashing scheme with an insertion probability* α*, we obtain* $a = \alpha \leq 1$*, and*
(i) $p(t)$ *converges to the stationary distribution* π^∞*, where*

$$\pi_k^\infty = \frac{(\alpha ch)^k}{k!} \Big/ \sum_{\ell=0}^{h} \frac{(\alpha ch)^\ell}{\ell!}, \quad k = 0, \ldots, h. \tag{5}$$

(ii) *the expected overflow fraction converges to* $(1 - \alpha) + \alpha \cdot \pi_h^\infty$*.*

Proof. The differential equations are the same as in the proof of Theorem 2 when replacing ch by αch, since α simply changes the arrival rate. The distribution results are then immediate. In addition, in the fixed-point equations, an arriving element either overflows immediately with probability $1-\alpha$, or checks with probability α a bucket that can be full with probability π_h^∞, hence the overflow equation follows as well. □

4. Overflow Lower Bound

Our objective is to find a lower bound on the expected overflow fraction γ of any $\langle a, d, c, h\rangle$ hashing scheme, when assuming a fluid model. We will study the simpler case with uniformly-distributed hash functions, as defined in Section 2. The more general case with several hash functions using different subtable-based distributions appears in Section 5.

The proof relies on the following result from [16]. Consider an Erlang blocking model with N servers, and suppose that the arrival rate depends on the system. Let $X(t)$ by the number of transmissions in progress at time t, and λ_k be the arrival rate when there are k transmissions in progress, $k = 0, 1, \ldots, N-1$. Then we have:

Lemma 1 (Theorem 4.2 in [16]). *For all increasing mappings* $f : \mathbb{R} \to \mathbb{R}$ *and for all* $t > 0$*,* $\mathbb{E}f(X(t))$ *is concave increasing as a function of* λ_k*, for* $k = 0, 1, \ldots, N-1$*.*

We use this lemma to prove the lower-bound result.

Theorem 5. *In the fluid model, under the assumptions above where only uniformly-distributed hash functions are used (see Definition 1), the optimal* *expected fixed-point overflow fraction* γ_{OPT} *in the* OPTIMAL DYNAMIC HASH TABLE PROBLEM *is lower-bounded by*

$$\gamma_{\text{LB}}^{\infty}(a) = 1 - a + a \cdot \frac{r^h}{h!} \Bigg/ \sum_{l=0}^{h} \frac{r^l}{l!}, \tag{6}$$

where $r = ach$.

Note again that the Erlang-B formula appears in the lower-bound on the overflow. This yields the following optimality result:

Theorem 6. *In the fluid model, the single-choice hashing scheme is optimal for every average number of memory accesses* a *in* $[0,1]$ *(and in particular for* $a = 1$*).*

PROOF. For the SINGLE scheme, there is a single hashed bucket per element, and it is accessed with probability α, therefore $a = \alpha$. For $a \leq 1$, we get

$$\gamma_{\text{LB}}^{\infty}(a) \overset{(a)}{=} (1-\alpha) + \alpha \cdot \frac{(\alpha ch)^h}{h!} \Bigg/ \sum_{l=0}^{h} \frac{(\alpha ch)^l}{l!} \overset{(b)}{=} \gamma_{\text{SINGLE}}^{\infty}$$

where (a) comes from Equation (6), $r = ach$ and $a = \alpha$, and (b) from Theorem 4. □

Example 1. *We illustrate the significance of the lower bound by considering a simple system with buckets of size* $h = 1$*, implying*

$$\gamma_{\text{LB}}^{\infty}(a) = 1 - a + a \cdot \frac{c \cdot a}{1 + c \cdot a} = 1 - \frac{a}{1 + c \cdot a}$$

.

In particular, for a load $c = 1$*, corresponding to the scaling case where the number of buckets is kept equal to the number of elements and therefore* $\lim_{n\to\infty} \frac{n}{m} = 1$*, we get*

$$\gamma_{\text{LB}}^{\infty}(a) = 1 - \frac{a}{1+a} = \frac{1}{1+a},$$

which shows that the lower-bound decreases slowly as $\Theta(1/a)$ *when the average number of memory accesses per insertion* a *increases.*

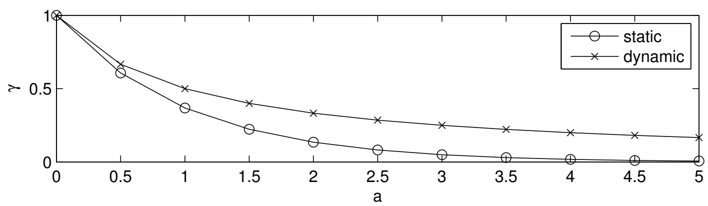


Figure 3: Expected overflow fraction as a function of the average memory access rate a.

For instance, to get a 1% *drop rate we need each element to access an average of at least* $a = 99$ *buckets. Of course, this is impossible to implement in high-speed networking devices. Thus, this lower bound is essentially an* impossibility result*, which shows that it is not easy to obtain efficient hash tables with deletions.*

Fig. 3 compares this drop rate lower-bound with the drop rate lower-bound in the static case, which is equal to e^{-a} [11]. As a increases, the figure shows how dynamic hash tables are significantly less efficient than their static counterparts.

5. Lower Bound with Multiple Hash-Function Distributions

We now consider a setting with a set $\mathcal{I}$ of $I = |\mathcal{I}|$ subtables, where subtable $i \in \mathcal{I}$ uses a fraction α^i of all buckets. We will allow for the d hash functions to use up to d different distributions $\{f_j\}_{1 \leq j \leq d}$ over the I subtables, where each distribution f_j assigns a probability f_j^i to subtable $i \in \mathcal{I}$, with $\sum_{i \in \mathcal{I}} f_j^i = 1$, and then uniformly picks buckets within each subtable (as defined in Section 2). We also assume that each distribution f_j is used by a fraction κ_j of the total memory accesses. Therefore, subtable i is accessed with a total probability of $\beta^i = \sum_{j=1}^{d} \kappa_j \cdot f_j^i$, with $\sum_{i \in \mathcal{I}} \beta^i = 1$. The following result establishes that the lower-bound is reached when the hash table is used in a uniform way, i.e. the probability β^i of accessing a subtable is equal to its fraction α^i in the table, and therefore the lower-bound is the same as established previously in Theorem 5.

Theorem 7. *In the fluid model with multiple distributions as defined above, the lower-bound* $\gamma_{\text{LB}}^{\infty}(a)$ *on the fixed-point overflow fraction is the same as the one with a unique uniform hash function, and is reached iff for all* $i \in [1, I]$, $\beta^i = \alpha^i$, *i.e. the weighted average of all distributions is uniform.*

6. A Multiple-Choice Hashing Scheme

We now introduce a natural extension to the single-choice hashing scheme that uses an ordered set of d hash functions $\mathcal{H} = \{H_1, \ldots, H_d\}$, such that all the hash functions are independent and uniformly-distributed. Upon inserting an element x, the scheme successively reads the buckets $H_1(x), \ldots, H_d(x)$ and places x in the first non-full bucket. If all these buckets are full, x is placed in the overflow list. To keep an average number of memory accesses per element of at most a, the algorithm attempts to insert x into the hash table with a probability α, otherwise it is directly placed in the overflow list. A detailed example appears in Appendix B.2.

We evaluate the performance of this scheme analytically using the fluid model. (Proof in Appendix A.6).

Theorem 8. *Assume the multiple-choice hashing scheme with a hashing probability α. Using the fluid-model fixed-point distribution π^∞,*

(i) π^∞ *satisfies* $\pi_k^\infty(a) = \dfrac{(ach)^k}{k!} \Big/ \displaystyle\sum_{l=0}^{h} \dfrac{(ach)^l}{l!}$, *for each* $k = 0, \ldots, h$;

(ii) *the average bucket access rate a satisfies the fixed-point equation* $a = \alpha \cdot \frac{1-\pi_h^\infty(a)^d}{1-\pi_h^\infty(a)}$;

(iii) *the expected overflow fraction is equal to the lower-bound, and is therefore* optimal, *for* $a \in [0, a^{co}]$, *where a^{co} satisfies the fixed-point equation* $a^{co} = \frac{1-\pi_h^\infty(a^{co})^d}{1-\pi_h^\infty(a^{co})}$.

The following example illustrates our results.

Example 2. *For the case where $h = 1$, solving the fixed-point equation yields* $a^{co} = \frac{2c-1+\sqrt{1+4c^2}}{2c}$. *Therefore, for a load of one element per bucket, i.e.* $c = \lim_{n\to\infty} \frac{n}{m} = 1$, *we get* $a^{co} = \frac{1+\sqrt{5}}{2} \approx 1.62$, *and the corresponding expected overflow fraction is* $\gamma_{\mathrm{LB}}^\infty(a^{co}) = 1.5 - \frac{\sqrt{5}}{2} \approx 38.2\%$. *Likewise, for a load of* $c = 0.1$, *we get* $a^{co} = \frac{-0.8+\sqrt{1+0.04}}{0.2} \approx 1.099$, *with the corresponding expected overflow fraction* $\gamma_{\mathrm{LB}}^\infty(a^{co}) \approx 0.98\%$.

7. Moving Back Elements

So far, we have found optimal schemes for a range of values of a, the average number of memory accesses per element. However, although optimal, the expected overflow fraction may still be too large.

In the literature, several solutions exist to reduce the drop rate (or collision probability) in a dynamic system. One such solution uses limited hash functions in order to be able to rebalance the hash table in case of deletion [17]. However, this approach gives up randomness, and the efficiency of a similar approach appears limited [7]. Another solution, based on the *second-chance* scheme [12], moves elements from one bucket to another by storing hints at each bucket [7]. These hints help to find another element stored in another bucket that can be moved upon the deletion. However, we found in simulations that this solution was less effective than our suggested scheme presented below for higher loads, while it was more effective for lower loads. Detailed simulation results are found in Section 8.

To reduce the overflow fraction, we suggest a scheme that allows moving elements back from the overflow list to the buckets upon a *deletion* operation[3]. This scheme can be combined with any insertion scheme.

7.1. Description

Our scheme, called the moving-back scheme (M-B), relies on a (binary) CAM. In general, a CAM stores keys in entries. Given some key k, a parallel lookup is performed over all entries and the index of the first (that is, highest priority) entry that contains k is returned from the CAM. In many cases, this index is later used in order to access in regular memory a direct-access array that contains the value associated with k. CAMs enable constant-time operations, however they are more expensive and consume more power than regular memory. It is a common practice to implement the overflow list in a CAM [12, 11, 1], relying on the fact that the number of elements in the overflow list is small.

Our scheme uses an auxiliary CAM, besides the primary CAM used to store the element of the overflow list: For each element x that is stored in the i-th entry of the primary CAM, we store the values $\{H_1(x), H_2(x), \ldots, H_d(x)\}$ in entries $d \cdot i, d \cdot i + 1, \ldots, d \cdot i + (d - 1)$ of the auxiliary CAM.

When an element is deleted from a bucket j that was previously full, we need to move an element x from the overflow list to bucket j such that j is the result of applying at least one of the hash-functions on x. We can locate such an element in constant time by querying the auxiliary CAM with key

[3] We also consider a scheme that works upon *insertion*, however the details are omitted due to lack of space; moving back elements upon deletion performs better in general.

j. Suppose the entry returned by the auxiliary CAM is ℓ, then x is located in entry $\lfloor \ell/d \rfloor$ of the primary CAM.

We note that upon moving an element back to the hash table, one should update the corresponding entries of the primary and auxiliary CAMs. An efficient way to update is to write the value $m+1$ in these entries, such that when a new element is inserted into the overflow list, one can query the auxiliary CAM with the value $m+1$ to decide in which entry (of the primary CAM) to put the new element.

7.2. Analysis

We first derive the exact expected overflow fraction in the case of the SINGLE scheme, and later provide an approximate model for the MULTIPLE scheme, which is confirmed by simulations.

Theorem 9. *Consider the* SINGLE *scheme with* M-B *for moving back elements from the TCAM and a symmetric insertion algorithm. The expected overflow fraction is given by:*

$$\gamma_{\text{LB}}(a) = 1 - \frac{1}{c} + \frac{1}{ch} e^{-ach} \sum_{k=0}^{h} (h-k) \frac{(ach)^k}{k!},$$

PROOF. Whenever a deletion occurs, the CAM device performs a lookup operation for any element that can be moved back to the bucket. Since every element has only one hash value, all elements that correspond to some bucket can be viewed as its own waiting list. Since the element we choose to delete follows a random process that is independent of any other random process in our system, and also the load is fixed, we conclude that the overflow fraction follows the static case exactly, which is given in [11]. □

Theorem 10. *Consider the* MULTIPLE *scheme with* M-B *for moving back elements from the CAM and a symmetric insertion algorithm. Let X_t^i is the occupancy of bucket i at step t and $P_0, \ldots, P_h$ be the equilibrium probabilities of the occupancy of each buffer. The probabilities can be modeled by the following Markov chain:*

$$P_{kj}^i = \Pr(X_t = j | X_{t-1} = k) = \begin{cases} g \cdot \frac{1}{m} & j = k+1, k < h \\ \frac{k}{n} & j = k-1, h > k > 0 \\ \frac{k}{n} \cdot e^{-\frac{\gamma chd}{P_h}} & j = k-1, k = h \end{cases}$$

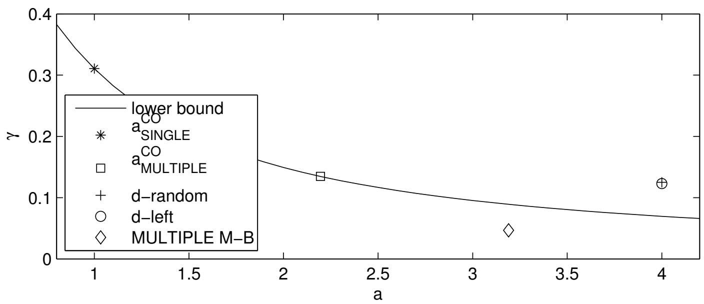


Figure 4: Expected overflow fraction as a function of a with $d = 4$, $h = 4$, $c = 1$.

where $g = \sum_{l=1}^{d} {P_h}^{l-1} = \frac{1-{P_h}^d}{1-P_h}$, *and the expected overflow fraction* γ *is given by* $\gamma = 1 - \frac{1}{ch} \cdot \sum_{i=0}^{h} i \cdot P_i$.

Proof. The Markov chain is the same as in the regular MULTIPLE scheme, except when an element is deleted from a full bucket. In this case, it is possible that one of the overflow elements in the CAM is moved back to the bucket. It is possible only in this case because all elements in the CAM have hashes to full buckets.

We now approximate the probability that none of the elements has a hash value to that bucket: The total number of hashes is $\gamma \cdot n \cdot d$, where all the hashes are to full buckets. The number of full buckets is $P_h \cdot m$. The probability that a single hash does not point to the specific bucket is $\frac{P_h \cdot m - 1}{P_h \cdot m} = 1 - \frac{1}{P_h \cdot m}$. And the probability that none of them points to the specific bucket is given by

$$\left(1 - \frac{1}{P_h \cdot m}\right)^{\gamma n d} \approx e^{-\frac{\gamma n d}{P_h \cdot m}} = e^{-\frac{\gamma c h d}{P_h}}.$$

Multiplying the above expression by the probability that one of the elements is picked for deletions in case the bucket is full yields the claimed Markov chain. □

8. Experimental Results

8.1. Simulations

Fig. 4 compares all the schemes by plotting the expected overflow fraction γ (Definition 2), as a function of the average number of memory accesses a.

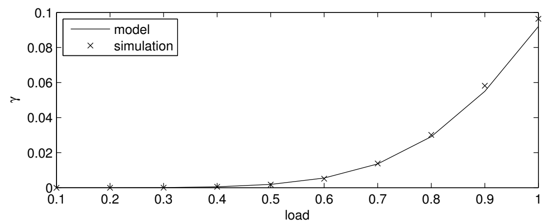


Figure 5: M-B with MULTIPLE scheme, for $h = 4$, $d = 2$ and different loads

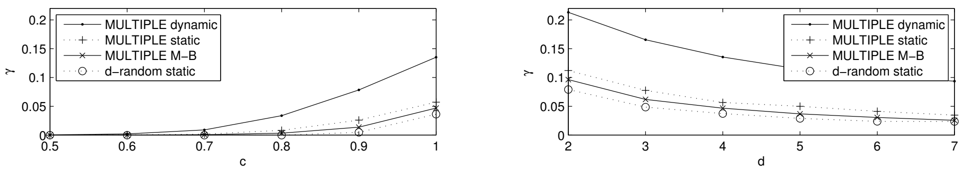
It was obtained with $d = 4$ choices, bucket size $h = 4$, $n = 4{,}096$ elements and $m = 1{,}024$ buckets, yielding a load $c = 1$.

The solid line plots the expected overflow fraction lower-bound $\gamma_{\text{LB}}(a)$ from Theorem 5. Simulations show that the proposed M-B scheme beats the lower bound with an expected overflow fraction of 4.6%, emphasizing the strength of this architecture. Of course, the lower bound does not apply to this case, since it moves back elements from the CAM.

As follows from Theorems 6 and 8, the expected overflow fractions $\gamma_{\text{SINGLE}}(a)$ and $\gamma_{\text{MULTIPLE}}(a)$ of the single-choice (SINGLE) and the multiple-choice (MULTIPLE) hashing schemes follow the lower-bound line, respectively until $a^{co}_{\text{SINGLE}} = 1$ with $\gamma_{\text{SINGLE}} = 31.1\%$, and $a^{co}_{\text{MULTIPLE}} = 2.195$ with $\gamma_{\text{MULTIPLE}} = 13.5\%$. Therefore, they are clearly optimal up to a certain point.

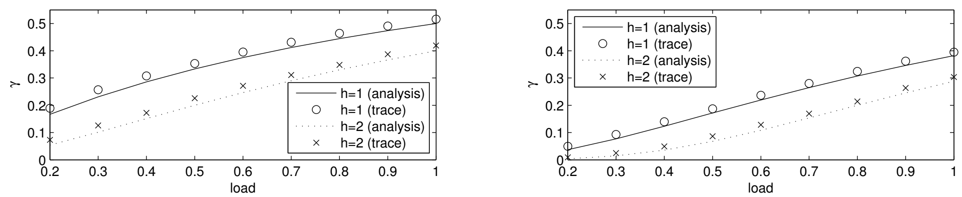
We also test our models from Section 7.2. Fig. 5 shows the accuracy of our M-B model. We ran simulations with $m = 1024$, $h = 4$, $d = 2$ and different loads. The maximum gap is for load $c = 1$ where our model predicts an expected overflow fraction of 9.20%, whereas simulations show an expected overflow fraction of 9.68%. For lower values of c, the model is much more accurate. For instance, for load $c = 0.5$, our model predicts an expected overflow fraction of 0.19% compared to an overflow fraction of 0.18% found via simulations.

We further evaluate the performance of our proposed M-B scheme. Quite surprisingly, when using the MULTIPLE scheme (of Section 6), the M-B scheme outperforms the *static* case of the MULTIPLE scheme (see Fig. 6), and performs similarly to the static d-random scheme (in the static case, d-random performs better than our multiple-choice scheme, albeit consuming significantly more energy [11]). This can be explained intuitively as follows: our moving-back strategy moves back an element to the only corresponding bucket which is not full; this is equivalent to inserting the element to the least occupied bucket as in the d-random hashing scheme.



(a) $h = 4$ and $d = 4$ (b) $h = 4$ and $c = 1$

Figure 6: Expected overflow fraction of the proposed moving-back (M-B) scheme (via simulations).



(a) SINGLE (b) MULTIPLE

Figure 7: Experiment using real-life traces and hash functions with SINGLE and MULTIPLE (d=2).

Finally, we compare the performance of our proposed M-B scheme with the performance of the hint-based scheme proposed in [7]. Note that our M-B scheme can be used with any insertion scheme. Thus, for fair comparison, since the hint-based scheme uses the second-chance scheme [12] for insertions, we also used the second-chance scheme for our proposed M-B scheme. We ran simulations with $m = 4096$, $h = 1$, $d = 4$ and different loads. As proposed in [7], the memory level sizes are exponentially decreasing with factor 2.

Fig. 8 shows that our M-B scheme is more effective than the hints-based scheme for higher loads, while it is less effective for lower loads. For instance, for a load of 0.6, the M-B and the hints-based schemes yield expected overflow fractions of 1.08% and 0.78%, respectively. For a load of 0.7, they yield 3.88% and 4.59%.

8.2. Experiments Using Real-Life Traces

We have also conducted experiments using real-life traces recorded on a single direction of an OC192 backbone link [18]. Our goal is to compare the average overflow fraction retrieved using our models for SINGLE and MULTIPLE with the corresponding overflow fraction when using a real hash function

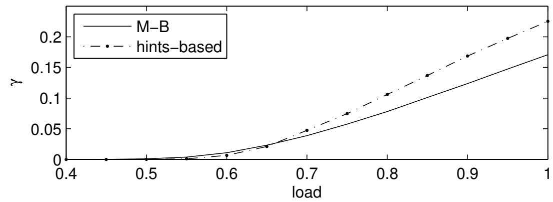


Figure 8: The M-B and the hints-based schemes, for $h = 4$, $d = 1$ and different loads.

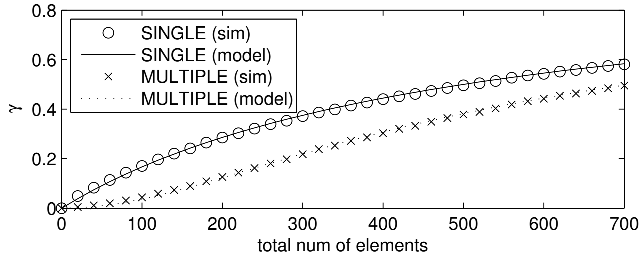


Figure 9: Marginal overflow fraction of 100 on-off flows with $m = 500$, $h = 1$ and $d = 2$

on a real-life trace. We used a 64-bit mix function [19] to implement two 16-bit hash functions. We used $m = 10{,}000$ buckets, and set a number of elements n as corresponding to various values of h and c. To keep a constant desired load, we alternated 100,000 times between an arrival (insertion) of a new TCP packet according to the trace, and the departure (deletion) of a random TCP packet. The hash functions were given the source and destination IP tuple as well as the sequence and acknowledgment numbers of the TCP packets. Therefore, the hash table stores the latest TCP packets, and can retrieve any needed packet based on its header. It can be used to monitor ongoing TCP flows, given a target number n of packets that are stored at any time. Its objective in our experiments was mainly to test the correctness of our model.

Fig. 7 shows that the results of our experiments are relatively close to our model. The maximum gap is for the SINGLE scheme with $h = 1$ and $c = 0.3$. Our model predicts an average overflow fraction of 23.08%, while the experiment yields 25.67%.

8.3. Experiments Using an On-off Arrival Model

We also consider a queueing model where at each step i, b_i elements arrive according to k independent on-off bursty flows of elements [20]; then, after the arrival phase, one element is randomly deleted. Therefore, the number of elements in the system keeps changing, contrarily to the previous models with a constant load.

Fig. 9 shows the marginal overflow fraction under the above queueing model with $k = 100$ on-off flows of elements. Each flow has rate $\rho = 0.0095$ and average burst size of 10 elements. The figure shows that, given the number of elements currently in the system, the marginal overflow fraction is approximately the one we found for the constant-load case, both for SINGLE and MULTIPLE.

Moreover, by the distribution of the number of elements in the system given by the queueing model, we are able to heuristically approximate the overall expected number of elements in the overflow list. More precisely, we take the sum-product of the queue size distribution by the distribution of the overflow fraction as a function of the load. In the case of SINGLE this model gives an expected number of overflow elements of 61.63, while simulations yield 61.41. Likewise, for MULTIPLE, we obtain 40.17 and 40.26, respectively. Therefore, this heuristic model proves quite accurate.

9. Conclusion

In this paper we demonstrated that, when the memory is bounded, dynamic schemes behave significantly worse than their static counterparts. This decrease in performance is inherent to the problem, as shown by our lower bounds.

Moreover, we considered two hashing schemes that we proved to be optimal: a single-choice hashing scheme that was used to demonstrate our approach and techniques, and a multiple-choice scheme that inserts the elements greedily.

However, due to the slow decrease of the lower bound, optimality may be insufficient for certain applications. Therefore, we suggested moving back elements from the overflow list as soon as a deletion occurs. We have shown through simulations that this strategy beats the lower bound of the dynamic case (where moving back elements is not allowed).

We also conducted an extensive experimental study to verify the accuracy of our model, the behavior of the models under realistic (rather than fully-random) hash functions, and under variable-load input models.

Acknowledgment

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Appendix A. Proofs

Appendix A.1. Proof of Theorem 1

We model the hash table using a *discrete-time Markov chain* that represents the occupancy X_t^i of an arbitrary bucket i at the end of time-slot t. We will see that this is possible because the process is memoryless from time-slot to time-slot, and because when conditioned on the occupancy of bucket i, its arrival and departure probabilities are independent of the states of the other buckets or of the overflow list.

At the end of each time-slot $t-1$, there are X_{t-1}^i elements in bucket i. Then, at the start of time-slot t, the element that departs is chosen uniformly at random out of the n elements in the system. Therefore, the probability that it belongs to one of the X_{t-1}^i elements in bucket i is $\frac{X_{t-1}^i}{n}$.

The element is then reinserted into the system. The probability that it is hashed by the uniformly-distributed hash function H into bucket i out of m buckets is $\frac{1}{m}$.

We can now build the state transition matrix. The bucket occupancy obviously increases iff there is no departure while there is an arrival, while it decreases iff there is a departure but no arrival. For $1 \leq j, k \leq h$, the transition probability from occupancy j to occupancy k is

$$\begin{aligned} P_{jk}^i &= \Pr\left(X_t^i = k | X_{t-1}^i = j\right) \\ &= \begin{cases} \frac{1}{m} \cdot \left(1 - \frac{j}{n}\right) & k = j+1, k \geq 1, \\ \frac{j}{n} \cdot \left(1 - \frac{1}{m}\right) & k = j-1, k \leq h-1, \\ \left(1 - \frac{j}{n}\right)\left(1 - \frac{1}{m}\right) + \frac{j}{n}\frac{1}{m} & k = j. \end{cases} \end{aligned}$$

The birth-death Markov chain is clearly irreducible, positive, recurrent and aperiodic. Therefore, it converges to its stationary distribution π^n. In addition, since the state transition matrix does not depend on i, by ergodicity, $p(t)$ also converges to π^n, as detailed in the theorem.

Lastly, bucket i overflows in a given time-slot t when it contains h elements, no element leaves bucket i, and an element arrives to bucket i. In addition, the probability that the element arriving at time t is sent to the overflow list is the sum of all individual bucket overflow probabilities. Therefore, by ergodicity, the total overflow probability at time-slot t converges to

$$\gamma^n_{\text{SINGLE}} = m \cdot \left(\pi^n_h \cdot \left(1 - \frac{h}{n}\right) \cdot \frac{1}{m} \right) = \pi^n_h \cdot \left(1 - \frac{h}{n}\right).$$

Since the mean time for which an element stays in the overflow list is n, we immediately get by Little's Law that the expected number of elements in the overflow list is $(\gamma^n_{\text{SINGLE}} \cdot n)$. $\square$

Appendix A.2. Proof of Theorem 2

In the fluid model, the departures from buckets of size $k \geq 1$ cause the fraction $p_k(t)$ to decrease at rate $k \cdot p_k(t)$, since each of the k elements departs at rate 1, and therefore the k elements depart at a total rate of k. Since the buckets of size k with a departing element have a new size $k-1$, the departures from such buckets increase in turn $p_{k-1}(t)$ at the same rate $k \cdot p_k(t)$.

Likewise, the arrivals to buckets of size $k < h$, which occur at rate $ch = \lim_{n\to\infty} \frac{n}{m}$, cause the fraction $p_k(t)$ to decrease at rate $ch \cdot p_k(t)$, and the fraction $p_{k+1}(t)$ to increase at the same rate.

Therefore, we obtain the following differential equation, which characterizes the birth-death process:

$$\frac{dp_k(t)}{dt} = \begin{cases} ch \cdot p_{k-1}(t) + (k+1)p_{k+1}(t) & \\ \qquad -(ch+k)p_k(t) & \text{for } k \in [1, h-1], \\ p_1(t) - ch \cdot p_0(t) & \text{for } k = 0, \\ ch \cdot p_{h-1}(t) - hp_h(t) & \text{for } k = h, \end{cases}$$

with $\sum_{k=0}^{h} p_k = 1$. Assume the system is initially empty, i.e. $p_k(0) = \mathbf{1}_{k=0}$.

Solving the differential equation above yields the stationary distribution of an $M/M/h/h$ loss system [15]. In addition, the drop rate in the fluid model is

$$\gamma^\infty_{\text{SINGLE}} = \pi^\infty_h,$$

following the well-known *Erlang-B formula*. Finally, since the differential equations are exactly those of the ergodic $M/M/h/h$ continuous-time Markov process, $p(t)$ converges to π^∞ [15, 21]. $\square$

Appendix A.3. Proof of Corollary 3

For each $n \in \mathbb{N}^* \cup \{\infty\}$, $\sum_{k=0}^{h} \pi_k^n = 1$ and $\pi_0^n > 0$, so for $k \in [0, h]$, π_k^n / π_0^n is defined and

$$\pi_k^n = \frac{\pi_k^n / \pi_0^n}{\sum_{l=0}^{h} \pi_l^n / \pi_0^n}.$$

Therefore, to prove the convergence of $\{\pi^n\}_{n \geq 1}$, which is a sequence of finite vectors, we only need to prove the point convergence of π_k^n / π_0^n to $\pi_k^\infty / \pi_0^\infty$. We get

$$\begin{aligned} \pi_k^n / \pi_0^n &= \binom{n}{k} \left(\frac{1}{m-1} \right)^k \\ &= \frac{1}{k!} \cdot \left(\frac{n}{m} \right)^k \cdot \frac{(1) \cdot \left(1 - \frac{1}{n}\right) \cdot \cdots \cdot \left(1 - \frac{k-1}{n}\right)}{\left(1 - \frac{1}{m}\right)^k} \\ &= \frac{1}{k!} \cdot (ch)^k \cdot (1 + o(1)) = \pi_k^\infty / \pi_0^\infty \cdot (1 + o(1)), \end{aligned}$$

which concludes the proof of the convergence of π^n to π^∞.

Lastly, $\gamma_{\text{SINGLE}}^\infty = \pi_h^\infty$ and $\gamma_{\text{SINGLE}}^n = \pi_h^n \cdot \left(1 - \frac{h}{n}\right)$. Since $\left(1 - \frac{h}{n}\right) = 1 + o(1)$, the convergence of γ_{SINGLE}^n to $\gamma_{\text{SINGLE}}^\infty$ follows. $\square$

Appendix A.4. Proof of Theorem 5

For any $k \in [1, h]$, let $p_k(t)$ denote the fraction of buckets of size k. As shown with the single-choice hashing scheme (SINGLE), in the fluid model, the departures from buckets of size $k \geq 1$ decrease the fraction $p_k(t)$ at rate $k \cdot p_k(t)$, and increase the fraction $p_{k-1}(t)$ at the same rate $k \cdot p_k(t)$.

Likewise, the element arrival rate before hashing is $ch = \lim_{n \to \infty} \frac{n}{m}$, and the hashing rate per element is a, therefore the hashing rate is ach. Let $r = ach$. Since elements can only decide to enter a bucket after hashing into it, we know that their post-hashing arrival rate to any bucket is bounded from above by r. Of course, the decision of whether to enter a bucket after hashing into it might depend on the bucket occupancy. Therefore, let $r_k(t)$ denote the average arrival rate to the fraction of buckets of size k at time t, with $0 \leq r_k(t) \leq r$. These arrivals cause the fraction $p_k(t)$ to decrease at rate $r_k(t) \cdot p_k(t)$, and the fraction $p_{k+1}(t)$ to increase at the same rate.

Combining departures and arrivals, we obtain the following differential equation characterizing the birth-death process:

$$\frac{dp_k(t)}{dt} = \begin{cases} r_{k-1}(t)p_{k-1}(t) + (k+1)p_{k+1}(t) & \\ \qquad -(r_k(t)+k)p_k(t) & \text{for } k \in [1, h-1], \\ p_1(t) - r_0(t)p_0(t) & \text{for } k = 0, \\ r_{h-1}(t)p_{h-1}(t) - hp_h(t) & \text{for } k = h, \end{cases}$$

with $\sum_{k=0}^{h} p_k = 1$ and $p_k(0) = \mathbf{1}_{k=0}$.

Consider a fixed point π of the birth-death process, i.e. assume that for any $i \in [0, h]$, $\frac{d\pi_k(t)}{dt} = 0$. Then the finite vector $(\pi_0(t), \ldots, \pi_h(t))$ is independent of t, and therefore the arrival rate $r_k(t)$ to a bucket of size k is independent of t as well. Denote this constant arrival rate as r_k, where $0 \leq r_k \leq r$. Solving the differential equation above yields the following balance equations: for each $k \in [1, h]$,

$$r_{k-1}\pi_{k-1} = k\pi_k, \tag{A.1}$$

with $\sum_{k=0}^{h} \pi_k = 1$. Therefore, π satisfies

$$\pi_k = \frac{\prod_{j=0}^{k-1} r_j}{k!} \cdot \pi_0 = \frac{\prod_{j=0}^{k-1} r_j}{k!} \Bigg/ \sum_{l=0}^{h} \frac{\prod_{j=0}^{l-1} r_j}{l!}.$$

Using Lemma 1, we find that the average bucket occupancy $\mathbb{E}X^{\pi}$ under π is upper-bounded by the average bucket occupancy $\mathbb{E}X^{\bar{\pi}}$ under $\bar{\pi}$, where $\bar{\pi}$ is the fixed-point distribution when $r_k = r$. This is because $f : X \to X$ is increasing [21], and $r_k \leq r$ for $k \in [0, h-1]$. Therefore,

$$\mathbb{E}X^{\pi} \leq \mathbb{E}X^{\bar{\pi}}. \tag{A.2}$$

Finally, note that without element losses, we would have the average occupancy equal to the average number of elements per bucket, i.e., $ch = \lim_{n\to\infty} \frac{n}{m}$. Therefore, the average fraction of lost elements is equal to

$$\begin{aligned} \gamma^{\infty} &= \frac{ch - \mathbb{E}X^{\pi}}{ch} \overset{(a)}{=} 1 - a \cdot \frac{\mathbb{E}X^{\pi}}{r} \overset{(b)}{\geq} 1 - a \cdot \frac{\mathbb{E}X^{\bar{\pi}}}{r} \\ &\overset{(c)}{=} 1 - a \cdot \frac{r \cdot (1 - \bar{\pi}_h)}{r} \overset{(d)}{=} 1 - a + a \cdot \frac{\frac{r^h}{h!}}{\sum_{l=0}^{h} \frac{r^l}{l!}}, \end{aligned}$$

where (a) uses $r = ach$, (b) relies on Equation (A.2), (c) uses a standard Erlang-B result [14, 15], and (d) comes from Equation (A.2) with $r_k = r$. $\square$

Appendix A.5. Proof of Theorem 7

As in the proof of Theorem 5, in each subtable i, we focus on the fixed-point distribution π^i, which satisfies

$$\pi_k^i = \frac{\prod_{j=0}^{k-1} r_j^i}{k!} \Big/ \sum_{l=0}^{h} \frac{\prod_{j=0}^{l-1} r_j^i}{l!},$$

with $r_j^i \leq \frac{\beta^i}{\alpha^i} \cdot r$. This is because the rate at which the elements check a bucket in subtable $i \in \mathcal{I}$ is proportional to the ratio of their probability β^i of picking subtable i by the proportional size α_i of subtable i. In addition, even if the elements check a bucket of size j, they can decide not to enter it. The rate r_j^i, at which they enter it, depends both on the size j and on the subtable i, although it needs to be upper-bounded by the rate $\frac{\beta^i}{\alpha^i} \cdot r$ at which they checked it.

From the proof of Theorem 5, in each subtable $i \in \mathcal{I}$, we know that the average occupancy is upper-bounded by the case in which we have equality

$$r_j^i = \frac{\beta^i}{\alpha^i} \cdot r.$$

We now want to find the vector β that maximizes the average occupancy of the whole system. Let $\bar{\pi}\left(\frac{\beta^i}{\alpha^i} r\right)$ denote the distribution that maximizes the average occupancy in subtable $i \in \mathcal{I}$, and define $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = \mathbb{E}X^{\bar{\pi}(x)}$. Then we want to find

$$\max \sum_{i \in \mathcal{I}} \alpha^i \cdot f\left(\frac{\beta^i}{\alpha^i} r\right) \qquad \text{s.t.} \quad \sum_{i \in \mathcal{I}} \alpha^i = 1, \quad \sum_{i \in \mathcal{I}} \beta^i = 1.$$

f is known to be strictly concave [22, 23, 24] (the concavity also follows from Lemma 1). Therefore

$$\sum_{i \in \mathcal{I}} \alpha^i \cdot f\left(\frac{\beta^i}{\alpha^i} r\right) \overset{(a)}{\leq} f\left(\sum_{i \in \mathcal{I}} \cdot \alpha^i \cdot \frac{\beta^i}{\alpha^i} r\right) = f\left(r \cdot \sum_{i \in \mathcal{I}} \beta^i\right) \overset{(b)}{=} f(r),$$

where (a) uses concavity and $\sum_{i \in \mathcal{I}} \alpha^i = 1$, and (b) uses $\sum_{i \in \mathcal{I}} \beta^i = 1$, with equality iff $\frac{\beta^i}{\alpha^i} r$ independent of i, i.e. $\beta^i = \alpha^i$ for all $i \in \mathcal{I}$, because $\sum_{i \in \mathcal{I}} \beta^i = \sum_{i \in \mathcal{I}} \alpha^i$. Finally, as in the proof of Theorem 5, the same upper bound on the average bucket occupancy corresponds to the same lower bound on the overflow fraction. $\square$

Appendix A.6. Proof of Theorem 8

For a given rate a, the differential equations are the same as for the single-choice hashing scheme (SINGLE) and satisfy the same fixed-point distribution (Theorem 6).

Let us now compute a. Following the definition of the fluid model, there is an independence in the following sense: Whenever an element arrives, the probability that it uses its l^{th} hash function H_l, for $1 \leq l \leq d$, is $\alpha \cdot (\pi_h^\infty)^{l-1}$; namely, the product of the probability α that it is not directly placed in the overflow list by the probability that the first $l-1$ hash functions mapped into full buckets. Then, the l^{th} trial is successful with probability $1-\pi_h^\infty$. Finally, there were d unsuccessful trials with probability $\alpha \cdot (\pi_h^\infty)^{d-1}$. Therefore, the average number of trials per element is:

$$a = \left(\sum_{l=1}^{d-1} l \cdot \alpha \cdot (\pi_h^\infty)^{l-1} (1 - \pi_h^\infty) \right) + d \cdot \alpha \cdot (\pi_h^\infty)^{d-1}$$

Using the general formula

$$\sum_{k=1}^{K} k x^{k-1} = \frac{1 - x^{K+1} - (1-x) \cdot (K+1) x^K}{(1-x)^2},$$

we get

$$\begin{aligned} a &= \alpha \left[\frac{1 - (\pi_h^\infty)^d - (1 - \pi_h^\infty) \cdot d \cdot (\pi_h^\infty)^{d-1}}{1 - \pi_h^\infty} + d \cdot (\pi_h^\infty)^{d-1} \right] \\ &= \alpha \cdot \frac{1 - (\pi_h^\infty)^d}{1 - \pi_h^\infty}. \end{aligned}$$

Finally, this can only hold for $\alpha \leq 1$; once we reach $\alpha = 1$, we obtain a^{co}_{MULTIPLE}. □

Appendix B. Examples

Appendix B.1. Single-Choice: Dynamic vs. Static

For the case where $h = 1$ the expected overflow fraction γ^n_{SINGLE} reduces to $\gamma^n_{\text{SINGLE}} = \frac{n-1}{m+n-1}$. Denoting the load $c = \frac{n}{m}$, $\gamma^n_{\text{SINGLE}} = \frac{c - \frac{1}{m}}{1 + c - \frac{1}{m}} \xrightarrow{m \to \infty} \gamma_{\text{SINGLE}} =$

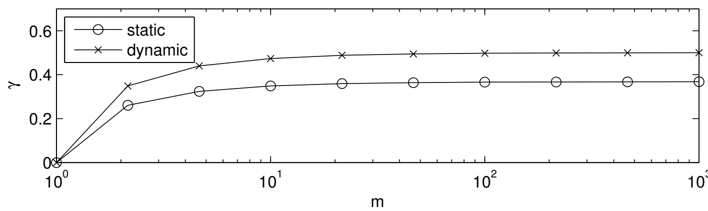

Figure B.10: Overflow fraction as a function of the number of buckets m.

$\frac{c}{1+c}$, where γ_{SINGLE} is the limit expected overflow fraction as we scale the system while keeping the load constant to c. For instance, for $c = 1$, we get

$$\gamma_{\text{SINGLE}} = 50\%. \tag{B.1}$$

In other words, when scaling the system with the same number of elements and buckets, we find that we asymptotically lose 50% of the elements.

Note that in such a scaling, we lose a fraction $\gamma^n_{\text{SINGLE}} = \frac{1-\frac{1}{m}}{2-\frac{1}{m}} = \frac{m-1}{2m-1}$ of the elements. This fraction corresponds for instance to no losses with $m = 1$; to 1/3 of the elements lost with $m = 2$; and to 40% of the elements lost with $m = 3$; the overflow fraction then continuing to increase monotonically and converge to γ_{SINGLE}.

Now we compare the *dynamic* overflow fraction γ^n_{SINGLE} with the *static* overflow fraction, denoted σ^n_{SINGLE}, given a bucket size of 1. First, assuming a load $c = 1$, the overflow fraction is equal to the fraction of unused buckets, because the number of elements is equal to the number of buckets, and buckets can contain at most one element. Therefore, since each element chooses a bucket uniformly at random, we get

$$\sigma^n_{\text{SINGLE}} = \left(1 - \frac{1}{m}\right)^m \xrightarrow{m\to\infty} \sigma_{\text{SINGLE}} = e^{-1}.$$

Thus, the static system has a clearly lower overflow fraction.

Fig. B.10 illustrates the overflow fraction as a fraction of m in both the static and dynamic cases. Clearly, the dynamic case always causes a higher overflow fraction. In addition, both converge fast from below to their limit values.

More generally, for any arbitrary load $c \leq 1$, the limit static overflow fraction is known [11] to be $\sigma_{\text{SINGLE}} = 1 - \frac{1-e^{-c}}{c}$. Therefore, as $c \to 0$, we asymptotically get $\gamma_{\text{SINGLE}} = c + O(c^2)$, and $\sigma_{\text{SINGLE}} = \frac{c}{2} + O(c^2)$, so for low

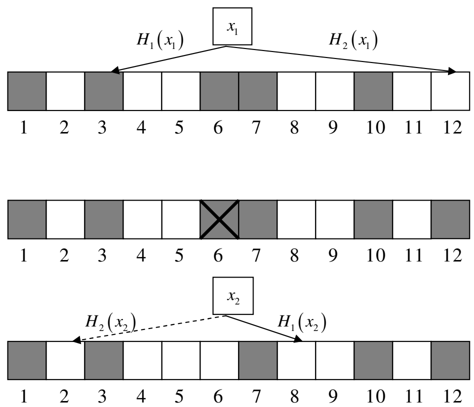

Figure B.11: Illustration of the MULTIPLE scheme

loads, the large dynamic system has about twice the overflow of the large static one.

Appendix B.2. Illustration of the MULTIPLE *Scheme*

Fig. B.11 illustrates the multiple-choice hashing scheme (MULTIPLE) with $m = 12$, $h = 1$, $d = 2$ and $q = 1$. We can see that element x_1 is initially mapped by H_1 to a full bucket. It is therefore mapped again by H_2, and inserted in an empty bucket. Then, the element in bucket number 6 is deleted. On the next step, element x_2 is directly inserted in an empty bucket, and therefore does not need a second memory access.

Vitae

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