Wave propagation with rotating intensity distributions

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General solutions representing rotations of intensity distributions around and along the propagation axis are derived for the paraxial wave equation. The formalism used is a key for understanding and synthesizing such waves as experimentally demonstrated. A necessary and sufficient condition for rigid rotation as well as limitations on the rotation rate are obtained. [S1063-651X(96)50507-6]

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Wave fields containing invariant features have recently stimulated the interest of the scientific community. Typical examples of such fields are Gaussian modes, Bessel beams [1], and wave fronts containing phase dislocations [2]. Bessel beams are solutions of the wave equation that propagate with invariant intensity. Phase dislocations are discontinuities of the phase in a wave front such that the circulation of the phase around its axis is an integral multiple of 2π. Thus, they determine lines of zero intensity in space. Experimental evidences of optical dislocations can be found, for example, in Refs. [3–5]. It was noted in Refs. [4,6] that, under certain circumstances, an array of dislocations nested in a Gaussian beam rotates by $\pi/2$ rad from the waist to the far field, expanding with the host beam.

The objective of this paper is to investigate general solutions having rotating intensity distributions around and along the propagation axis. We start by demonstrating that these solutions are easily obtained in terms of the superposition of Gauss-Laguerre (GL) modes. The rotation rate along the propagation is then derived and the set of all possible solutions presenting a specific total rotation angle is characterized. Finally, we analyze the limit of the rotation rate and present experimental results for optical beams.

Let a scalar wave be represented by the function

$$F(\mathbf{r},t)=u(\mathbf{r})\exp[i(kz-\omega t)], \tag{1}$$

where $\mathbf{r}=(\rho,\varphi,z)$ in cylindrical coordinates, ω is the angular frequency, and k is the wave number. The paraxial wave equation for propagation along the z axis is analogous to the Schrödinger equation of a free particle in two dimensions, where the z coordinate is replaced by the time variable [7,8]. This analogy allows us to use the formalism of quantum mechanics to analyze paraxial wave fields. Although the three-dimensional wave is stationary in time, we use time domain semantics to describe this evolution.

We seek solutions of the paraxial wave equation with scaled and rotated transversal intensity distributions. Our approach is to use a complete orthogonal set in which each basis function is stationary (except for scale) in the z direction, and is an eigenmode of rotation about the z axis. The GL function set satisfies both these requirements and the paraxial wave equation [8,9]. In analogy to quantum mechanics, dynamic behavior (in the z direction) is achieved by superposition of these modes. We found it convenient to write each GL mode as

$$u_{n,m}(\mathbf{r})=\langle \mathbf{r}|n,m\rangle=C_{n,m}G(\rho,z)R_{n,m}(\tilde{\rho})\Phi_m(\varphi)Z_n(z) \tag{2a}$$

where

$$G(\rho,z)=\frac{w_0}{w(z)}\exp[-\tilde{\rho}^{\,2}]\exp[ik\rho^2/2R(z)]\exp[-i\psi(z)], \tag{2b}$$

$$R_{n,m}(\tilde{\rho})=[\sqrt{2}\tilde{\rho}]^{|m|}L^{|m|}_{(n-|m|)/2}[2\tilde{\rho}^{\,2}], \tag{2c}$$

$$\Phi_m(\varphi)=\exp[im\varphi], \tag{2d}$$

$$Z_n(z)=\exp[-in\psi(z)], \tag{2e}$$

while $\tilde{\rho}=\rho/w(z)$ is the radial coordinate, scaled by the Gaussian spot size, which is given by

$$w(z)=w_0[1+(z/z_0)^2]^{1/2}, \tag{2f}$$

$$z_0=\pi w_0^2/\lambda \tag{2g}$$

is the Rayleigh length,

$$\psi(z)=\arctan(z/z_0) \tag{2h}$$

is the Gouy phase. The function (2b) is common to all modes, and comprises the radial Gaussian envelope of the beam, a Gouy phase, and a radial quadratic phase factor, with

$$R(z)=z[1+(z_0/z)^2] \tag{3}$$

being the radius of curvature of the wave front. $L^{|m|}_{(n-|m|)/2}$ are the generalized Laguerre polynomials, and the integers n,m obey the relation

$$n=|m|,\ |m|+2,\ |m|+4,\ \dots\ . \tag{4}$$

We use the factors $C_{n,m}$ to normalize the constant multiplicatives in (2c) around the axis (small $\tilde{\rho}$), leading to the expression

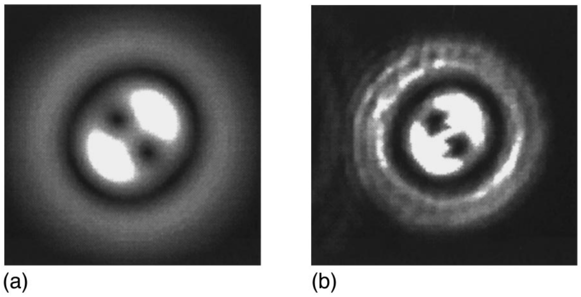

FIG. 1. (a) The theoretical intensity distribution at the waist and the far field. (b) Photograph of the far field.

$$C_{n,m}=\left[\left(\frac{n-|m|}{2}\right)!|m|!\right]\bigg/\left[\sqrt{2}^{|m|}\left(\frac{n+|m|}{2}\right)!\right]. \quad (5)$$

In Eq. (2a), the azimuthal dependence of each mode is given exclusively by $\Phi_m(\varphi)$ through the index m, while the dynamic behavior is determined by $Z_n(z)$ through the index n. To observe any kind of azimuthal change in transverse planes the wave must possess the following two characteristics.

(i) Anisotropy (no circular symmetry) about the z axis.
(ii) Nonstationary behavior in the z-direction.

The first characteristic can be achieved only by superposing modes with different m's, while the second is fulfilled only by superposing modes with different n's.

Let us examine a superposition of such modes. Assuming, without loss of generality, that $n_j \leqslant n_{j+1}$. The intensity distribution is given by

$$\begin{aligned} I(\mathbf{r}) &= \left|\sum_{j=1}^{M} a_j\langle \mathbf{r}|n_j,m_j\rangle\right|^2 \\ &= |G(\tilde{\rho})|^2\Bigg\{\sum_{j=1}^{M}|A_j|^2 R^2_{n_j,m_j}(\tilde{\rho}) \\ &\quad + \sum_{j=1}^{M}\sum_{p=j+1}^{M} 2|A_j||A_p|R_{n_j,m_j}(\tilde{\rho}) \\ &\quad \times R_{n_p,m_p}(\tilde{\rho})\cos[(m_j-m_p)\varphi-(n_j-n_p)\psi(z)-\vartheta_{jp}]\Bigg\} \end{aligned} \quad (6)$$

where the amplitude A_j comprises the complex amplitude a_j and the constant $C_{n,m}$ of mode j, while $\vartheta_{jp} \equiv [\arg(a_j) - \arg(a_p)]$. The first sum on the right hand side of Eq. (6) is isotropic about the axis and stationary in z. Each term in the second sum represents a wave rotating linearly with $\psi(z)$ at the local rotation rate

$$\dot{\varphi}_{jp}(z)=\left(\frac{d\varphi}{dz}\right)_{jp}=\frac{\Delta n_{jp}}{\Delta m_{jp}}\frac{d\psi(z)}{dz}, \quad (7)$$

where $\Delta n_{jp} \equiv n_j - n_p$ and $\Delta m_{jp} \equiv m_j - m_p$. Terms having $m_j = m_p$ are isotropic and they do not satisfy the first condition above. Therefore, they will not be considered further. If all the waves rotate at the same rate, scaled-rigid rotation will be observed. That is the case when $(d\varphi_{jp}/d\psi)=\text{const}$ for all j,p, leading to

$$\frac{n_{j+1}-n_j}{m_{j+1}-m_j}=\frac{\Delta n_j}{\Delta m_j}=\text{const}\equiv V, \quad j=1,2,\ldots,M-1. \quad (8)$$

If Eq. (8) is not fulfilled, additional ''harmonics'' will appear in the rotation. Hence Eq. (8) is a *necessary* and *sufficient* condition for rigid rotation of images in transverse planes. This result is in agreement with Ref. [10].

The total rotation from the waist ($z=0$), to the far field ($z=\infty$) is then $\Delta\varphi_{\text{total}}=V\pi/2$, as it is from $z=-\infty$ to the waist. Half of $\Delta\varphi_{\text{total}}$ is obtained at the Rayleigh distance.

We now study the limit of the achievable rotation rate with paraxial waves. As opposed to the spot size, which changes quadratically in the waist at a minimum rate (zero limit), the absolute azimuth changes linearly in the waist at its maximum rate [see Eqs. (2h) and (7)]. We assume the superposition of only two modes, since the rate is uniquely defined by their ratio V. In order to maximize the angular rate we assume $|\Delta m|=1$ and $n_1=0$, which leads to

$$\dot{\varphi}_{\max}(0)=n_2\lambda/\pi w_0^2. \quad (9)$$

Apparently, we can make this rate infinitely large by increasing the index n_2. However, the paraxial approximation imposes a trade-off between n_2 and w_0^2. To show this, we calculate the effective width of the beam as the standard deviation of the intensity distribution. We did so by using the analogs to the quantum-mechanical circular destruction and creation operators [11]. We thus get

$$\langle(\Delta x)^2\rangle=\frac{\langle u|X^2|u\rangle}{\langle u|u\rangle}=\frac{n+1}{4}w^2(z)\xrightarrow[z\gg z_0]{}\frac{n+1}{4}\left(\frac{\lambda z}{\pi w_0}\right)^2. \quad (10)$$

Hence the effective half-angular-beam-spread for a GL beam obeys

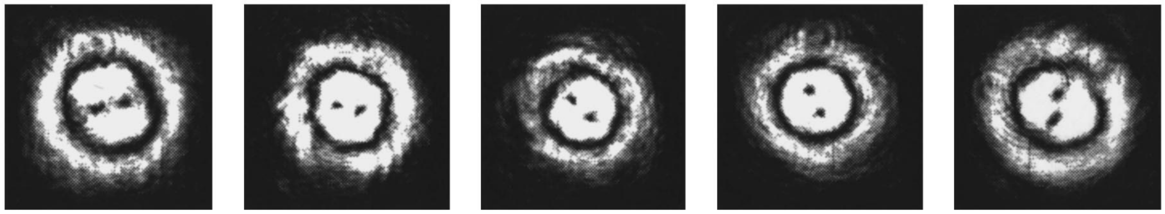

FIG. 2. Cross-sectional photographs along the z axis within the Rayleigh range. The middle photograph is taken near the waist.

$$\tan(\theta_{\text{beam}}) = \sqrt{n+1}\,\frac{\lambda}{\pi w_0}. \tag{11}$$

Using relation (11) in Eq. (9), we obtain the upper bound for the rotation rate as

$$\dot{\varphi}_{\max}(0) = \left(\frac{\pi}{\lambda}\right)\tan^2(\theta_{\text{beam}})\,\frac{n_2}{n_2+1} \xrightarrow[n_2\to\infty]{} \left(\frac{\pi}{\lambda}\right)\tan^2(\theta_{\text{beam}}). \tag{12}$$

The paraxial approximation limits θ_{beam} and thus we assume $\tan(\theta_{\text{beam}})\lesssim 1/2$. A similar criterion for the validity of the paraxial approximation was employed in Ref. [12]. Accordingly, we have

$$\dot{\varphi}_{\max}(0) \lesssim \frac{\pi}{4\lambda}. \tag{13}$$

Although the rotation rate increases asymptotically with n_2, it gets close to the limit of Eq. (12) even for small n_2. As a consequence of relation (12), if the beam is to pass effectively through an aperture D placed at $|z| = f \gg z_0$, then

$$\dot{\varphi}_{\max}(0) \approx \left(\frac{\pi}{\lambda}\right)\left(\frac{D}{2f}\right)^2. \tag{14}$$

We note that this rate is in accordance with the axial-spatial-frequency cutoff, which measures how fast the intensity can change along the direction of the optical axis [13].

As an example, we show experimental results of a beam with $\Delta\varphi_{\text{total}} = \pi$. The beam consists of a superposition of two GL modes:

$$|u\rangle = 6|4,2\rangle + |0,0\rangle. \tag{15}$$

We realized (15) by using a computer generated hologram. The beam had $w_0 = 0.2$ mm, using a He-Ne laser ($\lambda = 632.8$ nm). The hologram was positioned at $z = -2$ m, and had dimensions of 1×1 cm^2. It encoded the superposition of (15) and a plane wave at off axis angle of 0.35° on 1201×1201 binary pixels [14]. These figures were selected in order to obtain enough separation of the orders and exploit the resolution of the plotter, film, and camera, while keeping the physical dimensions of the beam and setup convenient for laboratory work. The theoretical transversal intensity distribution at the waist (and also at the far field) is shown in Fig. 1(a). The experimental photograph of the far field is given in Fig. 1(b). We show, in Fig. 2, cross-sectional photographs at several planes along the z axis within the Rayleigh range. The frame in the middle was taken approximately at the waist. Each image is rotated approximately by 30° relative to the adjacent ones. Note that the spot size is almost constant, while the rotation is substantial. The measured rotation angle along the propagation axis, compared to the theoretical curve, is presented in Fig. 3.

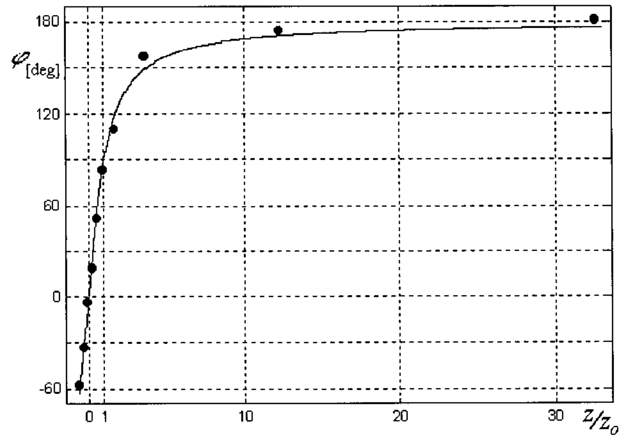

FIG. 3. The measured rotation angle along the propagation axis (black points), and the theoretical curve. The first five points to the left correspond to the frames of Fig. 2.

In conclusion, we characterized the set of all possible paraxial solutions presenting scaled-rotating intensity distributions along the propagation. We defined the rotation rate and observed that the total rotation angle from waist to far field is a rational multiple of $\pi/2$. The experimental demonstration showed good agreement with the theoretical predictions. The limits of the rotation rate in the paraxial regime were derived.

We further note that the total rotation, as well as the rotation rate, not only depend on the existence of phase dislocations (m numbers), but also on the envelope of the beam that hosts them (n numbers). The relation between rotations and self-imaging is presently under investigation.

It is worth nothing that the superposition of GL modes, having different optical frequencies, may lead to temporal rotations of the intensity distribution at fixed distances from the beam waist. This effect has been observed [5] in multimode beams generated by lasers.

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